Investigation of the moment of inertia of hollow cylinders

IB Physics Internal Assessment

Contents

Introduction

When I was 8 years old, my primary school science teacher performed an experiment in class. She showed us that a cylinder with a hole in it took longer to roll down a slope than a cylinder without a hole made from the same material. This intrigued me as I thought that the speed at which an object falls is independent of its mass, and thus the experiment lingers in my mind till this day. Therefore, for this IA, I will be investigating and researching the reasons behind this phenomenon and replicating the experiment with a much higher level of detail.

Research Question

How does the diameter of a hole in a hollow cylinder affect the cylinder's final velocity when rolling down a slope?

Background Information

The difference between the motion of a cube and a cylinder going down a slope is that the former would slip while the latter would roll.

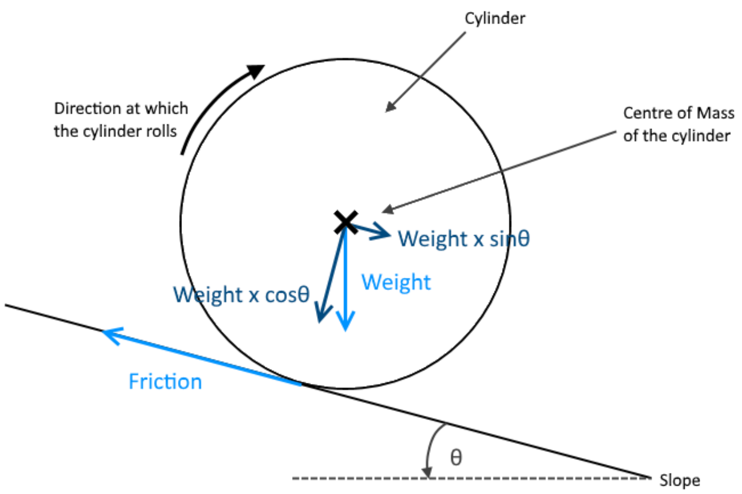

Figure 1 Free-body diagram of a cylinder rolling down a slope.

(Refer to Figure 1) When a cylinder is placed on a slope, two of the forces that act on it are its weight and the friction between the cylinder and the slope. Weight acts on the Centre of Mass of the cylinder, and it gives the cylinder a downward acceleration. However, friction acts on the surface of the cylinder instead of the Centre of Mass, and therefore giving the cylinder a torque, which is defined as force multiplied by the perpendicular distance from the line of action of the force to a point (Hamper 420), and in this case, friction multiplied by the perpendicular distance between the slope and the centre of mass of the cylinder. This is why the cylinder rolls down a slope instead of sliding down like a cube.

The rolling motion of the cylinder means that linear mechanics cannot be applied in its calculations. Rotational dynamics have to be applied instead. The equivalent of linear velocity in rotational dynamics is angular velocity, which is described by Equation 1.

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\Delta l}{r\Delta t} = \frac{v}{r} = 2\pi f$$

Equation 1 The relationship between angular velocity and frequency.

Where ω denotes the angular velocity of a rotating point, $\Delta\vartheta$ denotes the change in the angle and Δl denotes the change in arc length over Δt, r denotes the radius and v denotes the tangential velocity of the rotating point, and f denotes the frequency of the point.

The equivalent of mass in rotational dynamics is the moment of inertia, which is defined as an object's resistance to a change in its rotational motion (Homer 552). The moment of inertia of a point mass is given by Equation 2.

$$I = mr^2$$

Equation 2 The moment of inertia of a point mass.

Where I denotes the moment of inertia of the point mass, m denotes the mass of the point mass, and r denotes the distance of the point mass from its axis of rotation.

For an object with more than one point mass, the moment of inertia about a given axis can be calculated by adding the moments of inertia for each point mass (Homer 552). The summation of the moments of inertia of n point masses can be calculated as in Equation 3.

$$I = m_1{r_1}^2 + m_2{r_2}^2 + \cdots + m_n{r_n}^2 = \sum mr^2 = \int r^2 \, dm \qquad \text{(Pelcovits 225)}$$

Equation 3 The summation of the moments of inertia of n point masses and its mathematical abbreviations.

Therefore from Equation 3, we can deduce that the moment of inertia of a hollow cylinder with a total mass of M, a volume density of ρ, a height of h, an outer radius of R and an inner radius of R' can be calculated as the summation of the moments of inertia of infinitely small rings, or the integral of r^2 by m over the range of R' to R, as illustrated in Figure 2 and Equation 4.

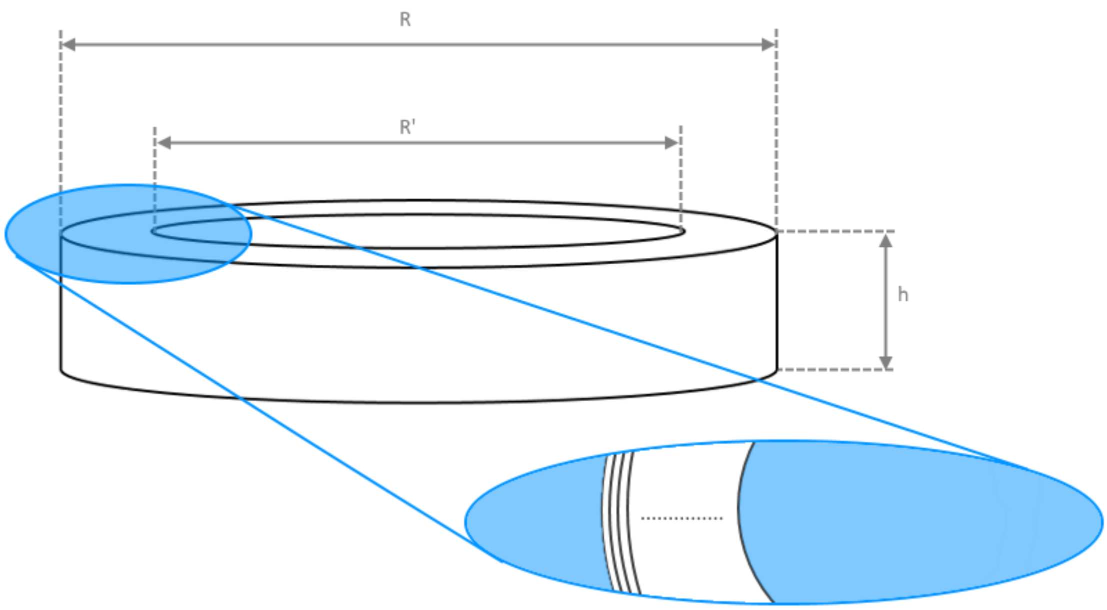

Figure 2 A hollow cylinder with the given conditions.

$$I = \int_{R'}^{R} r^2\, dm = \int_{R'}^{R} r^2 \cdot \rho \cdot dV = \int_{R'}^{R} r^2 \cdot \rho \cdot 2\pi r \cdot dr \cdot h = \rho 2\pi h \int_{R'}^{R} r^3\, dr = \rho 2\pi h \left(\frac{R^4}{4} - \frac{R'^4}{4}\right)$$

$$= \frac{\pi}{2}\rho h\left(R^4 - R'^4\right) = \frac{\pi}{2}\left[\frac{M}{\pi\left(R^2 - R'^2\right)h}\right] h\left(R^2 - R'^2\right)\left(R^2 + R'^2\right) = \frac{1}{2}M\left(R^2 + R'^2\right)$$

Equation 4 The derivation of the moment of inertia of a hollow cylinder with the given conditions.

With the equivalent of linear velocity and mass in rotational dynamics, we can obtain rotational kinetic energy. Any object rotating about an axis is said to have rotational kinetic energy (Giancoli 212). The rotational kinetic energy of a rigid rotating object, which is the sum of all the kinetic energy of the particles in the object, can be obtained as in Equation 5, along with using Equation 2.

$$E_{Kr} = \sum\left(\frac{1}{2}mv^2\right) = \sum\left(\frac{1}{2}mr^2\omega^2\right) = \frac{1}{2}\sum(mr^2)\omega^2 = \frac{1}{2}I\omega^2 \qquad \text{(Giancoli 213)}$$

Equation 5 The derivation of the rotational kinetic energy of a rigid rotating object.

An object that rotates while its centre of mass undergoes translational motion will have both translational and rotational kinetic energy (Giancoli 213). Therefore the total kinetic energy of a translating and rotating object can be given by Equation 6.

$$E_K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \qquad \text{(Homer 556)}$$

Equation 6 The total kinetic energy of a rotating and translating object.

Therefore, the final velocity of a hollow cylinder identical to that in Equation 4 released from rest on the top of a slope with a vertical height of *h*, can be calculated as in Equation 7.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}\left[\frac{1}{2}M\left(R^2 + R'^2\right)\right]\left(\frac{v}{R}\right)^2$$

$$4gh = 2v^2 + \left(R^2 + R'^2\right)\left(\frac{v^2}{R^2}\right)$$

$$v = \sqrt{\frac{4gh}{3 + \left(\frac{R'}{R}\right)^2}}$$

Equation 7 Derivation of the final velocity of a hollow cylinder in the given conditions.

Hypothesis

The moment of inertia of the cylinder increases as *R'* increases, which would increase rotational kinetic energy and decrease the translational kinetic energy. Therefore it is hypothesized that as *R'* of a hollow cylinder increases, and its final velocity in rolling down a slope will decrease.

Apparatus

Equipment	Quantity
Slope with a length of 150 cm	1
Lab Jack	1
Stand	1
Velocity Sensor (± 0.01 m s^{-1})	1
Data logger with Pasco Capstone	1
Hollow wooden cylinders with an outer diameter of 10 cm and inner diameters of 0, 1, 2,…, 8, 9 cm	1 of each
Ruler (± 0.05 cm)	1
Protractor (± 0.5°)	1

Table 1 Equipment list of the experiment, as well as the corresponding uncertainties of the equipment.

Variables

Types of Variables	Variables
Independent Variable	The inner radius of the hollow cylinder
Dependent Variable	The final velocity of the cylinder rolling down the slope

Table 2 Independent and dependent variables of the experiment.

Controlled Variables	Reason of control	Method of control
Outer radius of the hollow cylinder	The magnitude of the outer radius of the hollow cylinder will affect the moment of inertia of the hollow cylinder. The outer radius has to be kept constant in order to ensure a fair comparison between the different moments of inertia caused by the different inner radii of the hollow cylinders.	The hollow cylinders are all made out of a wooden cylinder with a diameter of 10 cm.
Height of the slope	The wooden cylinder will have a higher amount of gravitational potential energy if it is positioned higher above the ground. The height of the slope has to be kept constant for there to be a fair comparison between the final velocities of the wooden cylinders when they roll down the slope.	Since the same slope is used for all trials, the angle of the slope is kept constant to keep the height of the slope constant as well.

Materials of the cylinders and the slope	Different materials have different coefficients of frictions as well as different volume densities, which will affect the magnitude of the friction and the moment of inertia of the hollow cylinders. Thus the materials have to be kept constant to ensure a fair comparison between the different moments of inertia caused by the different inner radii of the hollow cylinders and the final velocities of the wooden cylinders when they roll down the slope.	Use the same slope for all trials and use the same type of wood to make all the wooden cylinders.

Table 3 Controlled variables of the experiment.

Experiment Procedures

1. Set up the apparatus as shown in Figure 3.
2. Turn on the velocity sensor and the data logger.
3. Let go of the cylinder at the top of the slope from rest.
4. Turn off the velocity sensor and the data logger after the cylinder rolled off the slope.
5. Repeat steps 1 to 4 for 3 times to obtain 3 trials for the experiment.
6. Repeat steps 1 to 5 for the remaining 9 cylinders.
7. After the experiment, collect and process the data for further analysis.

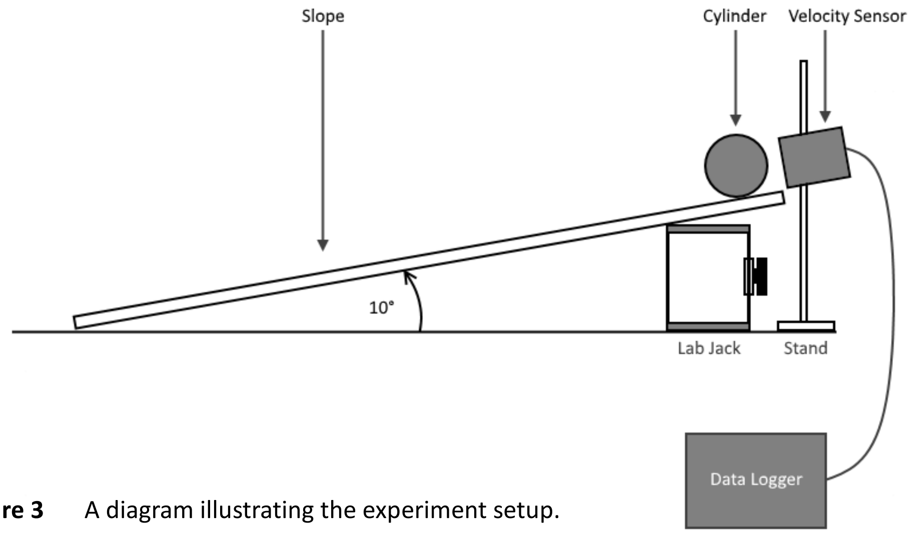

Figure 3 A diagram illustrating the experiment setup.

Safety, ethical and environmental concerns

Safety precautions such as wearing gloves and protective goggles must be taken during the creation of the cylinders. Furthermore, leftover wood from the construction of other products is used for the creation of the cylinders to prevent further cut-offs of trees upon taking ethical and environmental factors into consideration.

Data Collection, Processing and Analysis

Raw Qualitative Data

By observation, the final velocity at which the cylinders roll down the slope decreases as the diameter of the hole increases in the hollow cylinder.

Sample Calculations

The final velocities which the cylinders reach at the end of the slope is calculated by averaging the final 10 velocity data points recorded by the velocity sensor (Note that the velocity sensor records data at a frequency of 20 Hz). The data collected when the diameter of the hole of the hollow cylinder is 0 cm is used as an example.

Final velocity $(v) = \frac{1}{10} \times (1.42 + 1.55 + \cdots + 1.29 + 1.57) = 1.47 \text{ m s}^{-1}$ (2dp)
$\Delta v = \frac{1}{2} \times (1.57 - 1.35) = 0.11 \text{ m s}^{-1}$ (2dp)

Similarly, the final velocities of Trial 2 and Trial 3 can be calculated. The three final velocities calculated is then averaged to one value.

In Trial 1, $v = (1.47 \pm 0.11)\ \text{m s}^{-1}$; In Trial 2, $v = (1.48 \pm 0.10)\ \text{m s}^{-1}$; In Trial 3, $v = (1.45 \pm 0.12)\ \text{m s}^{-1}$
Averaged final velocity $(v_{avg}) = \frac{1}{3} \times (1.47 + 1.48 + 1.45) = 1.47 \text{ m s}^{-1}$ (2dp)
$\Delta v_{avg} = \frac{1}{2} \times (1.58 - 1.33) = 0.13 \text{ m s}^{-1}$ (2dp)

Similarly, using the above method, the final velocities and the averaged final velocities of all the cylinders can be calculated. The results are shown in Table 3.

Diameter of the hole (± 0.05 cm)	Final Velocities			Average final velocity (m s^{-1})
	Trial 1 (m s^{-1})	Trial 2 (m s^{-1})	Trial 3 (m s^{-1})	
0.00	1.47 ± 0.11	1.48 ± 0.10	1.45 ± 0.12	1.47 ± 0.13
1.00	1.45 ± 0.12	1.47 ± 0.11	1.47 ± 0.13	1.46 ± 0.14
2.00	1.41 ± 0.09	1.45 ± 0.15	1.44 ± 0.10	1.43 ± 0.15
3.00	1.43 ± 0.11	1.42 ± 0.13	1.43 ± 0.14	1.43 ± 0.14
4.00	1.36 ± 0.11	1.38 ± 0.12	1.37 ± 0.09	1.37 ± 0.13
5.00	1.35 ± 0.10	1.35 ± 0.10	1.33 ± 0.11	1.34 ± 0.12
6.00	1.34 ± 0.08	1.31 ± 0.14	1.32 ± 0.12	1.32 ± 0.14
7.00	1.25 ± 0.11	1.25 ± 0.13	1.27 ± 0.10	1.26 ± 0.13
8.00	1.21 ± 0.12	1.24 ± 0.11	1.24 ± 0.13	1.23 ± 0.14
9.00	1.12 ± 0.14	1.16 ± 0.09	1.17 ± 0.11	1.15 ± 0.15

Table 4 Processed raw quantitative data of the experiment.

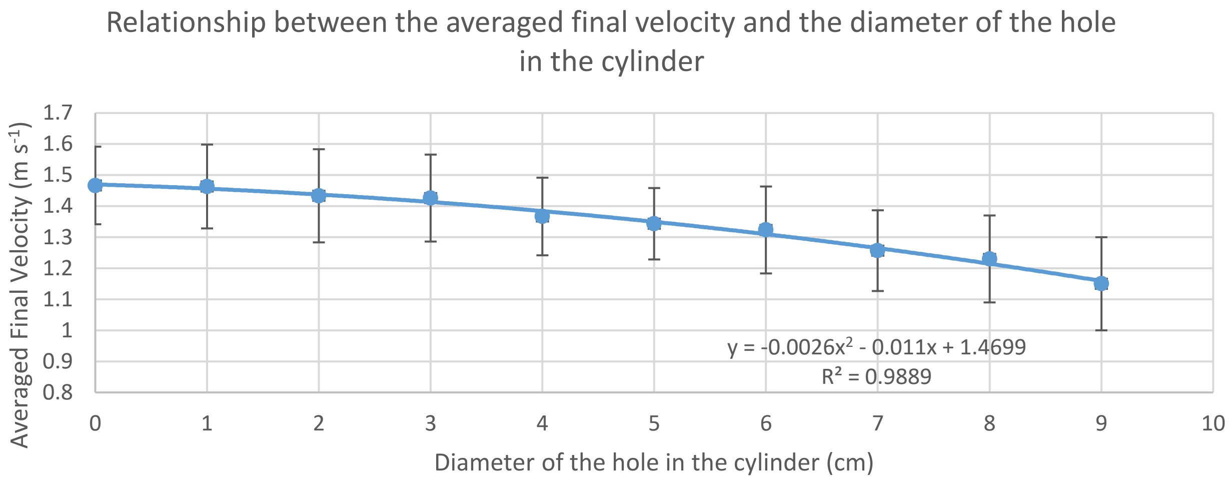

Figure 4 The relationship between the averaged final velocity and the diameter of the hole in the cylinder.

Experimental Graph Analysis

From Figure 4, we can see that as the diameter of the hole in the hollow cylinder increases, its averaged final velocity in rolling down the slope decreases with an increasing rate. A parabolic curve is used to fit the data as a parabolic curve resembles the trend suggested by the data, and a curve with an equation of $y = -0.0026x^2 - 0.011x + 1.4699$ is yielded.

Linearization of the Relationship

From Equation 4, the expression of the final velocity of a cylinder rolling down a slope is derived. The expression showed that given that the acceleration due to gravity and the vertical height of the slope remain constant, the final velocity is proportional to the inverse of the root of the square of the ratio between the inner radius and the outer radius of the hollow cylinder added by 3 (This will be named as the Ratio Factor in this report). The proportionality is shown in Equation 8.

$$v = \sqrt{\frac{4gh}{3+\left(\frac{R'}{R}\right)^2}} \Rightarrow v \propto \frac{1}{\sqrt{3+\left(\frac{R'}{R}\right)^2}}$$

Equation 8 The proportionality of the final velocity to the Ratio Factor.

To see if the data obtained from the experiment fits with the theoretical model of this experiment, the data obtained will be plot against the Ratio Factor of the cylinders. If the data produces a linear relationship with the Ratio Factor, it means that the data obtained fits the model.

The Ratio Factor of the hollow cylinders have to be calculated. The hollow cylinder with a hole with a diameter of 9 cm is used as an example.

$$\text{Ratio Factor } (P) = \frac{1}{\sqrt{3+\left(\frac{9/2}{10/2}\right)^2}} = 0.512 \quad \text{(3sf)}$$

$$\frac{\Delta P}{P} = \frac{1}{2} \times \left[2 \times \left(\frac{\Delta R}{R} + \frac{\Delta R'}{R'}\right)\right]$$

$$\Delta P = 0.005$$

Similarly, the Ratio Factors of all the other cylinders can be calculated. The results are shown in Table 1 in Appendix 2.

The average final velocity is then graphed against the Ratio Factor of each of the cylinders. The relationship is shown in Figure 5.

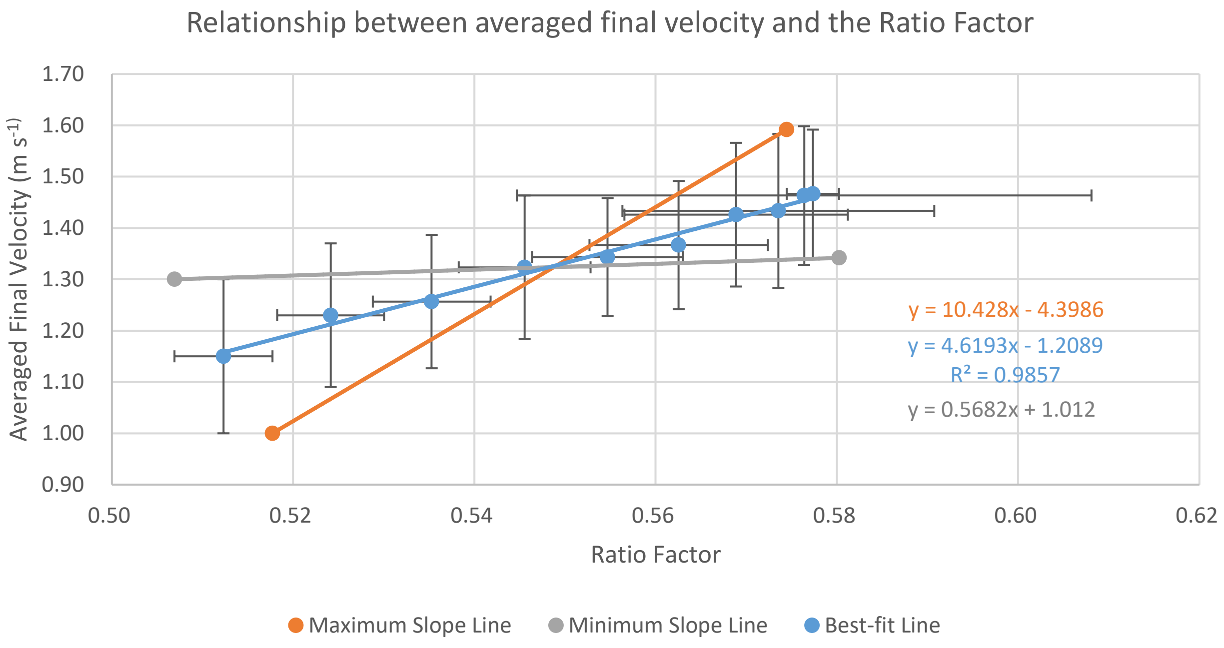

Figure 5 The relationship between averaged final velocity and the Ratio Factor.

Linearized Graph Analysis

From Figure 5, we can see that as the Ratio Factor increases, the final velocity increases linearly. A very high R^2 value of 0.9875 is produced. Furthermore, we can see that the slope of the best-fit line falls between the maximum slope line and the minimum slope line (0.5682 < 4.6193 < 10.428). This means that the data obtained from the experiment fit the theoretical model of the experiment.

Comparison with Theory

From Equation 7, the theoretical final velocity at which a hollow cylinder rolls down a slope can be calculated. The hollow cylinder with a hole of a diameter of 9 cm is used as an example.

$$v = \sqrt{\frac{4gh}{3+\left(\frac{R'}{R}\right)^2}} = \sqrt{\frac{4\times 9.81\times 1.50\sin(10^\circ)}{3+\left(\frac{9/2}{10/2}\right)^2}} = 1.64 \text{ m s}^{-1} \quad (3\text{sf})$$

($\because$ The slope has a length of 150 cm and is put at 10° to the table, $\therefore h = 1.50\sin(10^\circ)$)

Similarly, the theoretical final velocities of all the other cylinders can be calculated. The results are shown in Table 1 in Appendix 3.

To compare the calculated theoretical values with the experimental data, the experimental and theoretical relationships between the final velocities, the diameters of the hole in the hollow cylinders and the Ratio Factors will be graphed. These are shown in Figures 6 and 7.

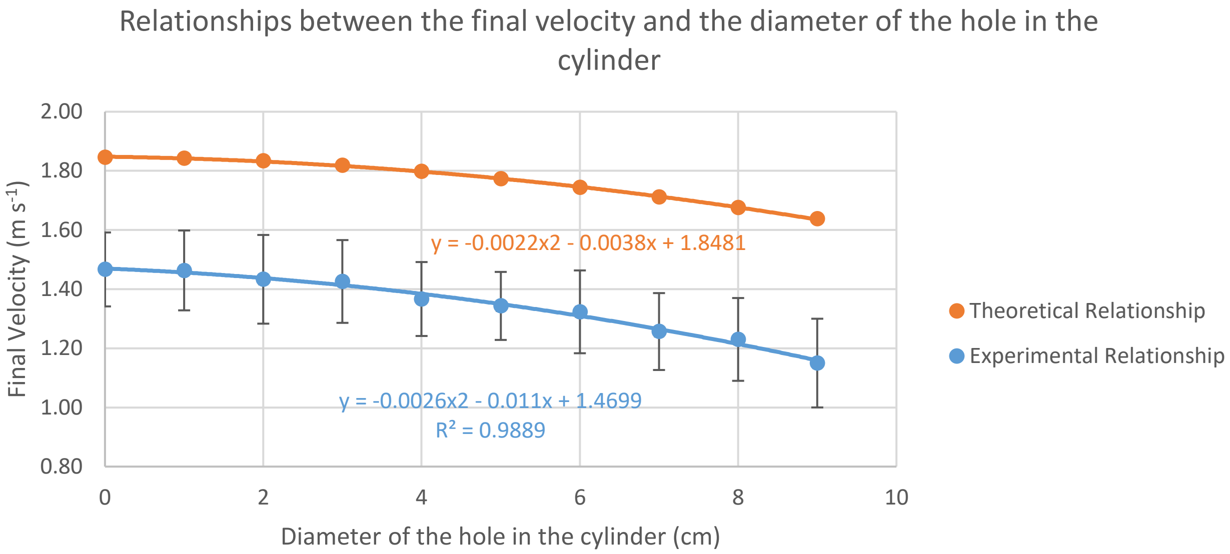

Figure 6 Theoretical and Experimental relationship between final velocity and the diameter of the hole in the cylinder.

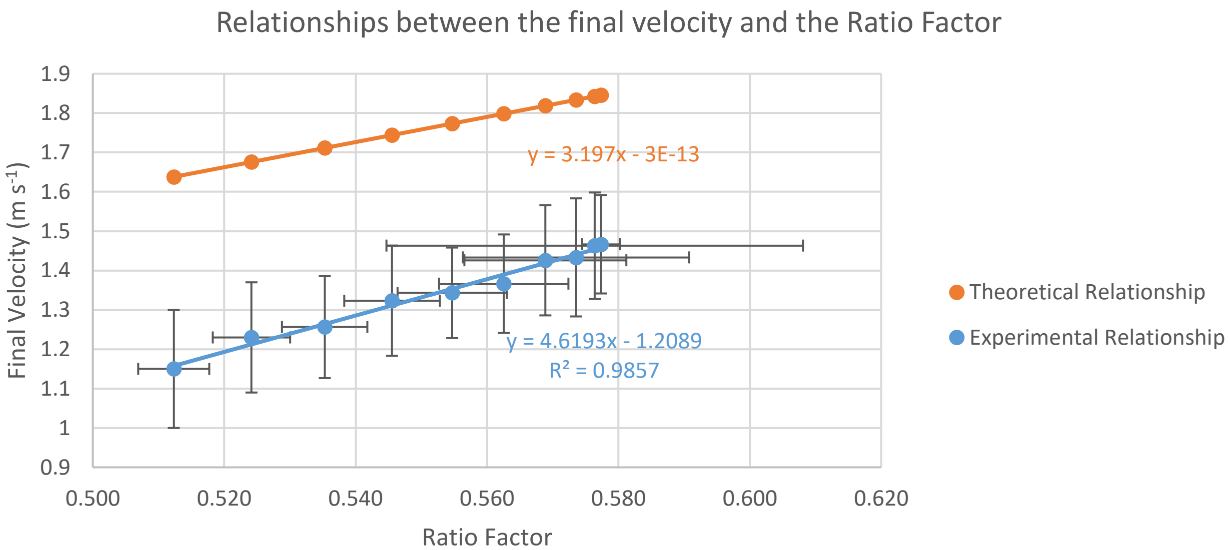

Figure 7 Theoretical and Experimental relationship between final velocity and the Ratio Factor.

Experimental and Theoretical Graphs Comparison

From Figures 6 and 7, we can see that the theoretical values calculated have a generally higher value than the experimental values. However, the discrepancies between both relationships are small. Both theoretical relationships has the same trend as the experimental relationships, meaning that the experimental data obtained are relatively accurate.

Conclusion and Evaluation

Conclusion

This investigation aims to answer the question "How does the diameter of a hole in a hollow cylinder affect the cylinder's final velocity when rolling down a slope?" It is hypothesized that as the diameter of a hollow cylinder increases, and its final velocity in rolling down a slope would decrease.

Through experimentation and data processing, the results showed that as the diameter of the hole in a hollow wooden cylinder increases, its final velocity in rolling down a slope decreases with an increasing rate, and the relationship can be represented by the question $y = -0.0026x^2 - 0.011x + 1.4699$ in Figure 4, verifying the hypothesis. This phenomenon can be explained by the moment of inertia of the hollow wooden cylinders increasing as the inner radius of the hollow wooden cylinders increase. A hollow cylinder with a higher moment of inertia will have higher rotational kinetic energy, which in turn decreases its translational kinetic energy, causing it to have a lower velocity in rolling down a slope when compared with a cylinder with a lower moment of inertia.

To show that the experimental data obtained fits the mathematical model of the experiment, the experimental data is linearized by plotting the final velocities against the Ratio Factors of the wooden cylinders. Upon linearization, the results showed that as the Ratio Factor increased, the final velocity increases linearly, and the relationship can be represented by the equation $y = 4.6193x - 1.2089$ with a very high R^2 value of 0.9875 in Figure 5. Furthermore, the slope of the best-fit line falls between the maximum slope line and the minimum slope line. This means that the data obtained from the experiment fits the theoretical model of the experiment.

The graphs of the obtained relationships are plotted with the theoretical mathematical relationships for comparison in Figures 6 and 7. A slight discrepancy between the two relationships are seen in both cases as the theoretical values are higher than the experimental values. However, the theoretical relationship and the experimental relationship both showed the same trends. The slight discrepancy along with the experimental values showing the same trend as the theoretical values means that the experimental data obtained is relatively accurate. Therefore, this experiment is a success in showing the relationship between the diameter of the hole in a hollow cylinder and its final velocity in rolling down a slope.

Evaluation

Random Error	Effect on experiment	Method of Improvement
Accidental incorrect measurements of the final velocities of the wooden cylinders	It is noted that in some trials, the wooden cylinders do not travel down the slope in an exactly straight path. Therefore, the velocity sensor may not have measured the actual final velocity of the wooden cylinders, and the actual final velocity of the wooden cylinders may be higher than the measured value.	Create a straight track on the slope that is aligned with the velocity sensor for the wooden cylinders to roll on.

Table 5 Source of random error in the experiment.

Systematic Error	Effect on experiment	Method of Improvement
Energy loss due to air resistance	There is air resistance opposing the motion of the cylinder as it goes down the slope, and energy is lost in opposing the air resistance, resulting in the final velocities obtained through the experiment being less than theoretical velocities, as shown in Figures 7 and 8. But as energy loss due to air resistance is almost impossible to account for with secondary school lab equipment, and as the cylinders are moving at relatively slow speeds such that air resistance will not affect the results dramatically, we assumed that there is no air resistance opposing the motion of the cylinder as it goes down the slope, and that there is no energy loss due to air resistance.	Account for the energy lost due to air resistance in the calculations for the theoretical final velocities of the cylinders in rolling down the slope to obtain a more accurate theoretical value, and create a better comparison between the theoretical and experimental results.
Slipping of the cylinders when they moved down the slope	Slipping occurs when the instantaneous velocity at the point of contact between a rolling body and the medium supporting it is not zero (Hecht 306). Slipping would result in the cylinders sliding and rotating down the slope at the same time, causing errors in the experimental velocities. However, it is assumed that no slipping occurs when the cylinders roll down the slope in this experiment.	Use materials that have a higher coefficient of friction with the slope to construct the hollow cylinders to reduce the chance of slipping.
Incorrect calibrations of the equipment	The velocity sensor may have incorrect calibrations and zero errors, which may cause the measured values of the final velocity to be either higher or lower than the actual values.	Use two velocity sensors to measure the final velocity of the cylinders in rolling down the slope to see if the measured values of one deviates from the other.

Table 6 Sources of systematic error in the experiment.

Extensions and Further Investigations

Extensions

This investigation found the relationship between the diameter of the hole in a hollow cylinder and its final velocity in rolling down a slope. The results can be significant in wheel design, as the ideal tire and wheel rim radius can be determined for different vehicles, such as bicycles, cars, trucks and aircraft. One on hand, bicycle wheels, for example, do not have to spin at a large angular velocity and have to be light-weight for commercial use. On the other hand, aircraft wheels, for example, have to have high durability and have to spin at a large angular velocity during take offs and landings. Through using similar experiment and data processing methods in this experiment, the ideal tire and rim radius, and thus the ideal tire designs for different vehicles can be determined.

Further Investigations

Firstly, further investigations can be conducted with the mentioned improvements in the Evaluation applied to the experiment. From the improved experiment, we can draw new and more accurate results on the final velocity of the cylinder in rolling down a slope, as well as create a more accurate comparison between the theoretical and experimental results.

Secondly, further investigations can also be conducted to see how the different materials used to make the cylinder affect its final velocity in rolling down a slope. Different materials have different volume densities, which may have an effect on the moment of inertia of the cylinder, thus affecting the final velocity of the cylinder in rolling down the slope. Therefore, a relationship between the volume density of the cylinders and their final velocities in rolling down a slope can be drawn and analysed.

Thirdly, since this experiment can be said to be investigating how different diameters of the hole in the hollow cylinder affects in rotation by its centre, another further investigation can be investigating how the diameter of the hole in the hollow cylinder affect its maximum rotational velocity when rotated by its diameter by a constant power source. The results of the two investigations can then be compared with each other to see if the relationship between the maximum rotational velocity by the centre of the cylinder and the diameter of the hole is similar to the relationship between the maximum rotational velocity by the diameter of the cylinder and the diameter of the hole.

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Appendix 1

Photos

The following are photos taken during the experiment.

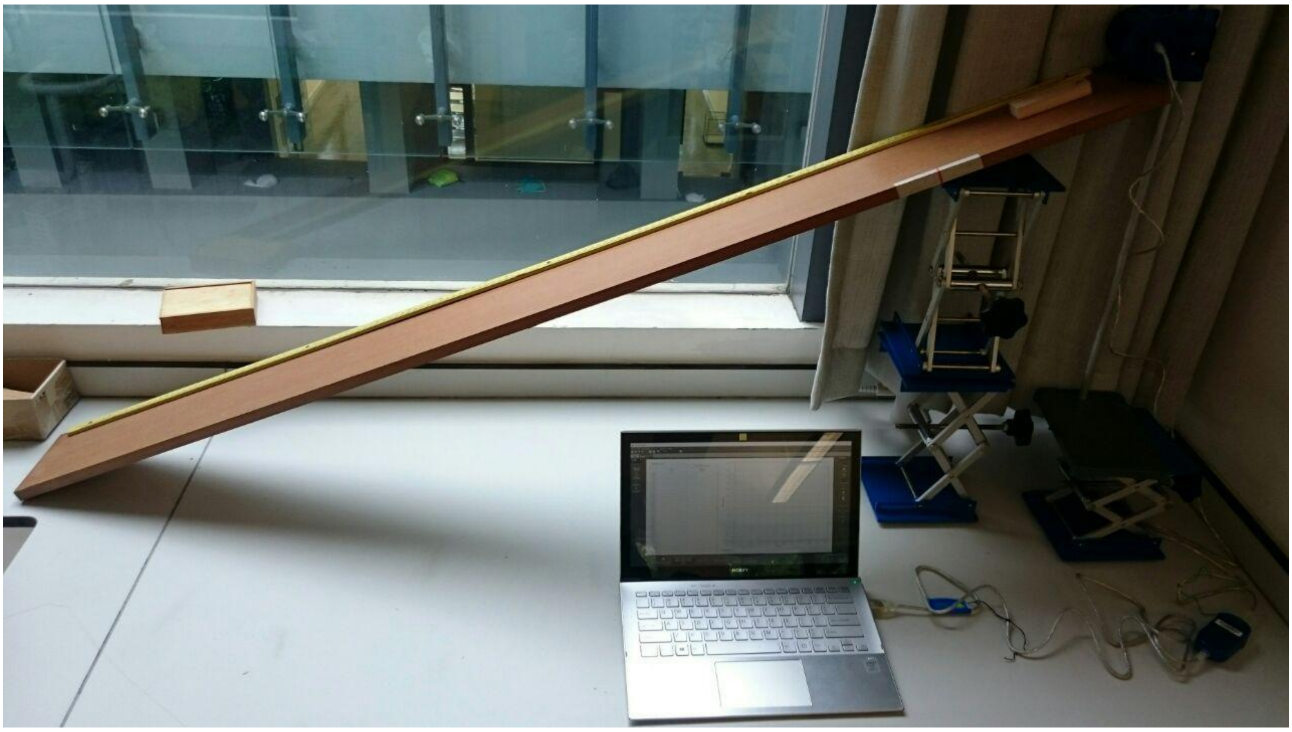

Photo 1 Photo of the experiment setup.

Photo 2 Photo of the wooden cylinders.

Appendix 2

Ratio Factor for different diameters of the hole in the hollow cylinders

Diameter of the hole in the hollow cylinder (± 0.05 cm)	**Ratio Factor**
0.00	0.577 ± 0.003
1.00	0.576 ± 0.032
2.00	0.574 ± 0.017
3.00	0.569 ± 0.012
4.00	0.563 ± 0.010
5.00	0.555 ± 0.008
6.00	0.546 ± 0.007
7.00	0.535 ± 0.006
8.00	0.524 ± 0.006
9.00	0.512 ± 0.005

Table 1 The Ratio Factors for different diameters of the hole in the hollow cylinder.

Appendix 3

Theoretical Final Velocities

Diameter of the hole in the hollow cylinder (cm)	Theoretical Final Velocity (ms^{-1})
0	1.85
1	1.84
2	1.83
3	1.82
4	1.80
5	1.77
6	1.74
7	1.71
8	1.68
9	1.64

Table 1 Calculated theoretical final velocities.